ON KNOT INVARIANTS WHICH ARE NOT OF FINITE TYPE

Theodore Stanford * and Rolland Trapp **

Abstract. We observe that most known results of the form "v is not a finite-type invariant" follow from two basic theorems. Among those invariants which are not of finite type, we discuss examples which are "ft-independent" and examples which are not. We introduce (n, q)-finite invariants, which are generalizations of finite-type invariants based on Fox's (n, q) congruence classes of knots.

1. How not to be of finite type.

Vassiliev [21] defined a family of knot invariants based on the study of singular knots. Gusarov [9] independently described the same family of invariants using very different methods and these invariants, now often called *finite-type invariants*, have been derived and analyzed from a number of different points of view. There is now quite a large body of literature. The widespread interest is due mostly to the following theorem, various parts of which were proved by various authors. See Birman [3] for an exposition and general proof. See also Bar-Natan [2], Birman and Lin [4], and Gusarov [8]. By "quantum knot polynomial" we mean the Jones polynomial or one of its many generalizations.

Theorem A. Let $P_K(t)$ be a quantum knot polynomial, and let $a_n(K)$ be the x^n coefficient of the power series obtained by substituting $t = e^x$ into $P_K(t)$. Then a_n is a finite-type invariant of order $\leq n$.

The invariants in Theorem A will all be rational-valued. However, we take "finite-type invariant" to mean any knot invariant v taking values in an abelian group A, such that there exists a positive integer n with $v(K) = 0$ for any singular knot K with more than n singularities. The value of v on a singular knot is determined by taking the difference between the positive and negative resolutions of the singularity in the standard way.

Theorem A gives us many finite-type invariants, and an immediate and natural question becomes whether there are invariants which are not of finite type. It is well-known that all finite-type invariants are determined by those which take values in $\mathbb{Z}$ and those which take values in $\mathbb{Z}_m$ (for all m), though in fact all known examples are determined by the $\mathbb{Z}$-valued invariants. The set of finite-type invariants taking values in all these abelian groups is easily seen to be countable, whereas the set of all $\mathbb{Z}$-valued knot invariants is uncountable. So in fact there are many invariants which are not of finite type. One could also ask whether there are invariants which are not determined by finite-type invariants. This is the same question as whether there exists a pair of knots, all of whose finite-type invariants are equal. The answer to this question is unknown.

Along similar lines, one may ask whether a particular known and studied knot invariant is of finite type. A number of standard knot invariants have been shown not to be of finite type by different authors using various techniques. See Altschuler [1], Birman and Lin [4], Dean [5], Eisermann [6], Ng [13], and Trapp [20]. Most of these results follow from one or both of the following two theorems:

Theorem B. No invariant taking a unique value on the unknot is of finite type. That is, if $v(K) = v(\text{unknot})$ implies that K is the unknot, then v is not a finite-type invariant.

Proof: There are now a number of constructions yielding nontrivial knots with trivial invariants up to any fixed order n. See for example Gusarov [9], Lin [11], Ohyama [14], Stanford [18]. □

It follows from Theorem B that none of the following are finite-type invariants: crossing number, unknotting number, bridge number, tunnel number, braid index, genus, free genus, Seifert genus, stick number, and crookedness.

Theorem C. Any knot invariant v for which connected sums cannot cancel is not of finite type. That is, if v is a knot invariant such that there exists a knot K with $v(K\#K') \neq v(\text{unknot})$ for all knots K', then v is not a finite-type invariant.

Proof: Gusarov [9] showed that for any knot K and any integer n, there exists a knot K' such that $v(K\#K') = v(\text{unknot})$ for any finite-type invariant of order $< n$. Other proofs of this result may be found in Habiro [10], Ng [13], and Stanford [19]. □

Let $P_K(t)$ be any knot polynomial with integer coefficients which has the property that $P_{K\#K'} = CP_K(t)P_{K'}(t)$, where $C \in \mathbb{Z}[t^{\pm 1}]$ is some fixed constant. (Most of the quantum knot polynomials fit this description.) Then by Theorem C none of the following are finite-type invariants: $P_K(t)$ as an element of the abelian group of Laurent polynomials; $P_K(n)$, where n is any integer such that there exists a knot K with $P_K(n) \neq \pm 1$; $P_K(\alpha)$, for any non-algebraic $\alpha \in \mathbb{C}$; the span of $P_K(t)$; the first and last coefficients of $P_K(x)$; the degrees of the first and last coefficients of $P_K(t)$.

Let G be a nonabelian group, and let v be a knot invariant with some value a such that $v(K) = a$ implies the existence of an nonabelian homomorphism from the fundamental group of the complement of K into G. By Theorem C, v is not a finite-type invariant, because if such a homomorphism exists for K then it is easy to see that it exists for $K\#K'$ no matter what K' is. If G is finite, then the number of homomorphisms from the group of K into G is not a finite-type invariant. Nor is the determinant of a knot (the Alexander polynomial evaluated at $t = -1$), since if this is not 1 then there exist nontrivial representations of the group of K into some dihedral group.

2. Ft independence.

A well-known invariant which is not covered by either Theorem B or Theorem C is the signature of a knot. The signature was originally shown not to be of finite type by Dean [5] and Trapp [20]. Ng [13] has proved a much more general theorem:

Theorem D. The only knot concordance invariant which is also of finite type is the Arf invariant. In fact, for any knots K_1 and K_2 with equal Arf invariant, and for any positive integer n, there exists a knot K_3 with the same concordance class as K_1, and such that $v(K_2) = v(K_3)$ for any finite-type invariant of order $< n$.

Theorem D inspires a definition: we shall say that a knot invariant v is *ft-independent* if for any set of finite-type invariants $v_1, v_2, \ldots v_m$, the values of $v_i(K)$ put no restrictions on the values of $v(K)$. More precisely, v is ft-independent if for any knots K_1 and K_2 and for any positive integer n there exists a knot K_3 such that $w(K_3) = w(K_2)$ for any finite-type invariant w of order $< n$, and such that $v(K_3) = v(K_1)$.

Thus, by Theorem D, any knot concordance invariant which is independent of the Arf invariant is ft-independent. However, many of the invariants that we have listed as being not of finite-type as a result of Theorems B and C are also clearly not ft-independent. Crossing number, for example, is not ft-independent. Choose a positive integer m and a finite-type invariant v. If $v(K) \neq v(K')$ for all knots K' of crossing number $\leq m$, then clearly the crossing number of K is greater than m. Thus the value of v places some restrictions on the possible crossing number of a knot.

Another invariant which is neither finite-type nor ft-independent is the degree of the Conway polynomial. If the $(2n)$th coefficient is nonzero then the degree of the Conway polynomial is at least $2n$. The coefficients of the Conway polynomial are themselves finite-type invariants (Bar-Natan [2]), the $t = e^x$ substitution being unnecessary because the Conway polynomial doesn't have terms of negative degree. Hence nonzero values for these invariants place lower bounds on the degree of the Conway polynomial. Since the degree of the Conway polynomial (divided by 2) is a lower bound for the genus of a knot, it follows that the genus is also not ft-independent.

Clearly, if $P_K(t)$ is a quantum knot polynomial, then, as an invariant taking values in the abelian group of Laurent polynomials with integer coefficients, it is not ft-independent because of Theorem A.

One example of an ft-independent invariant which is not a concordance invariant is the number of prime factors of a knot. Given a knot K and a positive integer n, one can form the connected sum of K with any number of knots whose invariants are trivial up to order n so as to produce a K' with an arbitrarily large number of prime factors. Going the other way, it was shown in Stanford [18] that given a knot K and a positive integer n, there exists a prime knot K' such that $v(K) = v(K')$ for every finite-type invariant of order $\leq n$.

3. (n, q)-finite invariants.

In this section we use Fox's notion of (n, q) congruence of knots to generalize finite-type invariants. Before defining (n, q)-finite invariants, however, we recall the definition of (n, q)-congruence classes of links (see Fox [7], Nakanishi and Suzuki [12], Przytycki [15]). We consider only links in $\mathbf{S}^3$, and follow the definition given in Przytycki [16].

Given a link L and a disk D^2 which L intersects transversely, let $U = \partial D^2$ and $q = |lk(L, U)|$. Note that the complement of U in $\mathbf{S}^3$ is a solid torus T with meridinal disk D^2. A $t_{2,q}$ move on L is the restriction to L of a Dehn twist in T on the disk D^2. Thus

a $t_{2,q}$ move has the effect of cutting L along D^2, inserting a full twist and reglueing. A $t_{2n,q}$ move is just the result of n Dehn twists on D^2, i.e. $t_{2n,q} = t_{2,q}^n$. Two links L_1, L_2 are then called *congruent modulo* (n,q), denoted $L_1 \equiv L_2 (\mathrm{mod}\ n, q)$, if one can obtain L_2 from L_1 by a sequence of $t_{2n,q'}^{\pm 1}$ moves together with isotopies, where the q' can vary but are required to be multiples of q.

The Alexander Module is a good tool for constructing invariants of (n,q) congruence, and this is done by Fox as well as Nakanishi and Suzuki. In Przytycki [16], the $t_{2n,q}$ moves are considered as generalizations of crossing changes. Generalized unknotting numbers are defined, and lower bounds for these unknotting numbers are given. Analogously, we will define generalized finite-type invariants by replacing the crossing changes in the Vassiliev skein relation by the more general $t_{2n,q}$ moves.

Recall the diagramatic definition of finite-type invariants. A link invariant f is of order $\leq m$ if for each link diagram D and every collection $(a_1, \ldots, a_{m+1})$ of $m+1$ crossings, the alternating sum

$$\sum_{\vec{i}} (-1)^{|\vec{i}|} f(D_{\vec{i}}) \tag{3.1}$$

vanishes, where $\vec{i}$ is in $\mathbb{Z}_2^{m+1}$, $|\vec{i}|$ is the number of nonzero coordinates in $\vec{i}$, and $D_{\vec{i}}$ is the diagram D with crossing a_j changed whenever the jth coordinate of $\vec{i}$ is one. Thinking of $t_{2n,q}$ moves as generalized crossing changes, we define f to be (n,q)-*finite of order* $\leq m$ if for all links L and collections $D_1^2, \ldots, D_{m+1}^2$ of mutually disjoint disks, with $\partial D_i^2 = U_i$, satisfying $q_j = \mathrm{lk}(U_j, L) \equiv 0 \bmod q$, the alternating sum

$$\sum_{\vec{i}} (-1)^{|\vec{i}|} f(L_{\vec{i}}) \tag{3.2}$$

vanishes, where $L_{\vec{i}}$ is obtained by t_{2n,q_j} moves on each U_j for which the jth coordinate of $\vec{i}$ is one.

Theorem E. Let $q \geq 0$. If f is a finite-type invariant of order $\leq m$, then f is an (n,q)-finite invariant of order $\leq m$. If f is $(1,q)$-finite of order $\leq m$, then f is finite-type of order $\leq m$.

Proof. Let f be a $(1,q)$-finite invariant of order $\leq m$. In our definition of (n,q)-finite, the q_j are allowed to be 0. Since crossing changes can be obtained by a Dehn twist on a disk whose boundary has linking number zero with the link, any sum (3.1) can be written in the form of (3.2) (with $n = 1$). Thus f is of finite type.

Now suppose that f is a finite-type invariant of order $\leq m$. Note that the difference $f(L) - f(t_{2,q}(L))$ can be realized as a sum of values of f on singular links. This follows directly from Theorem 3.1 of Stanford [17], but the intuitive idea is that the full twist in $t_{2,q}(L)$ can be undone by crossing changes, and thus any sum (3.2) may be written as a sum of expressions (3.1). This argument works for $t_{2n,q}$ as well, or else one can note that an (n,q)-finite invariant is (kn,q)-finite for all $k \in \mathbb{Z}^+$, because $t_{2kn,q} = t_{2n,q}^k$. $\square$

Thus (n,q)-finite invariants include finite-type invariants. It remains to show that the inclusion is proper. It is easy to see that invariants of (n,q)-congruence classes of links

are exactly the (n, q)-finite invariants of order 0 (in the same way that the only finite-type invariants of order 0 are those which are invariant under crossing changes, namely, anything which depends only on the number of components in the link). Thus we need only find an invariant of (n, q)-congruence which is not of finite type. The following is a direct result of Lemma 2.6b of Przytycki [16], which states that the number of $2n$-colorings of a link is invariant under certain $t_{2n,q}$ moves, and of Theorem C above, which implies that the number of colorings of a knot is not a finite-type invariant:

Theorem F. The number of $2n$ colorings of a link is an order 0 (n, q)-finite invariant for any even q. Thus there are order 0 (n, q)-finite invariants which are not of finite type.

Remark: Nakanishi and Suzuki define two links to be $q - congruent\ modulo\ n$ if they are related by a sequence of $t_{2n,q}^{\pm 1}$ moves, thus making the restriction that $lk(U, L) = q$. One can similarly refine the definition of (n, q)-finite invariants. Lemma 2.6a of Przytycki [15] can be applied to this refined notion of finite invariants. The reason it doesn't apply directly to (n, q)-finite invariants as defined above is that one could have $lk(U, L) = 0$. This implies that the geometric linking number is even and violates the hypotheses of Lemma 2.6a in [15].

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Mathematics Department
United States Naval Academy
572 Holloway Road
Annapolis, MD 21402

`stanford@nadn.navy.mil`

Mathematics Department
California State University, San Bernardino
San Bernardino, CA 92407

`trapp@math.csusb.edu`